JOURNAL OF MATHEMATICAL PSYCHOLOGY 37, 629–639 (1993)

Theoretical Note

Algebraic Representation of Additive Structures with an Infinite Number of Components

RONALD HÜBNER

Universität Braunschweig

AND

REINHARD SUCK

Universität Osnabrück

Representation theorems for additive conjoint structures with an infinite number of components were first developed by T. C. Koopmans (1960, *Econometrica* **28**, 287–309) for modelling dynamic decision behavior. However, he proved his theorems within a topological framework. Here, the theorems are adapted to and proved within a more general algebraic framework. Moreover, while Koopmans provided a representation only for bounded component sequences, here the representation is extended to unbounded sequences satisfying polynomial growth. © 1993 Academic Press, Inc.

Additive conjoint structures with an infinte number of components were introduced by Koopmans (1960) to model dynamic decision behavior. He derived representations for preference orderings of consumption programs (see also Koopmans, Diamond, and Williamson, 1964; Diamond, 1965; Koopmans, 1972). Recently, Hübner (1989a) showed that such structures are not only useful in modelling other dynamic behavior as well, but that they are also valuable in combining axiomatic measurement theory and linear system theory. By successfully modelling loudness adaption he demonstrated that his ideas are even suitable for system identification (Hübner, 1989b).

A disadvantage of Koopmans' results is that they are developed within the topological framework of Debreu (1960). It is not only that this formalism is uncommon for psychologically motivated measurement theories, but, as has been mentioned by Krantz, Luce, Suppes, and Tversky (1971) and recently emphasized by Wakker (1988), the algebraic approach of Krantz *et al.* (1971) is more general than the topological; i.e., the algebraic approach is applicable to more cases.

Reprint requests should be sent to Ronald Hübner, at Institut für Psychologie, Technische Universität Braunschweig, Spielmannstr. 19, D-38106 Braunschweig, Germany.

0022-2496/93 $5.00

Moreover, Koopmans starts his work by already assuming real valued representations of the components and taking advantage of the strong properties of the reals. Here, we start with qualitative axioms leading to such representations of the components. This approach is advantageous with respect to questions concerning the testability of structures. Thus, there are several reasons making it desirable to adapt Koopmans' concepts to an algebraic formalism.

Although Hübner (1989a) reformulated the main theorem of Koopmans (1960) in algebraic terms, an algebraic proof was not given. In this paper we present this proof. Moreover, Koopmans only furnished a representation for bounded sequences. Here, we also derive a representation for unbounded sequences satisfying *polynomial growth.*

In order to adapt Koopmans' results to an algebraic framework, some of the well-known concepts of finite additive conjoint measurement must be reformulated. Other definitions remain in the form used in n-component structures (cf. Krantz *et al.*, 1971).

DEFINITION 1. (a) Let $I=\{1,2,...\}$. A binary relation $\succsim$ on the infinite Cartesian product $A_1\times A_2\times\cdots$ *induces* for each $z\in A_1\times A_2\times\cdots$ and all $M\subset I$ a relation $\succsim_M^z$ by

$$(a_1,a_2,...)\succsim_M^z(b_1,b_2,...) \quad\text{iff}\quad (a_1^{z,M},a_2^{z,M},...)\succsim(b_1^{z,M},b_2^{z,M},...),$$

where

$$a_i^{z,M}:=\begin{cases}a_i & \text{if } i\in M\\ z_i & \text{if } i\notin M.\end{cases}$$

(*b*) $\succsim$ is *independent* iff for any finite or cofinite (i.e., $I-M$ finite) set $M\subset I$ and all z the induced relation $\succsim_M^z$ is independent of z.

Obviously, if $\succsim$ is a weak order all induced $\succsim_M^z$ are also weak orders, and in the case of an independent $\succsim$ for all M the orders $\succsim_M^z$ coincide and are denoted by $\succsim_M$. Instead of $\succsim_{\{i\}}$ and $\succsim_{\{ij\}}$ we simply write $\succsim_i$ and $\succsim_{ij}$, respectively.

DEFINITION 2. A binary relation $\succsim$ on $A_1\times A_2\times\cdots$ satisfies *restricted solvability* iff for all $i\in I$, whenever there exist $\bar{b}_i,\underline{b}_i\in A_i$ such that

$$(b_1,...,\bar{b}_i,...)\succsim(a_1,...,a_i,...)\succsim(b_i,...,\underline{b}_i,...),$$

then there exists a $b_i\in A_i$, such that

$$(b_1,...,b_i,...)\sim(a_1,...,a_i,...).$$

DEFINITION 3. Let $\succsim$ be an independent, connected, and transitive relation on $A_1\times A_2\times\cdots$, that is, an independent weak order. For any set S of consecutive integers a set $\{x_{1i}\mid x_{1i}\in A_1,\ i\in S\}$ is a standard sequence (of component 1), iff there are α, β in some other component A_k, $k \in N$, $k \neq 1$, such that not $(\alpha \sim_k \beta)$ and for all i, $i+1 \in M$, $x_{1i}\alpha \sim_{1k} x_{1i+1}\beta$. A parallel definition holds for the other components.

DEFINITION 4. Let $\succsim$ be a binary relation on $A_1 \times A_2 \times \cdots$. The component A_i is called *essential* iff there are $a, b \in A_i$ and for some $j \neq i$, $j \in I$, there exists $p \in A_j$, such that not $ap \sim_{ij} bp$.

DEFINITION 5. (a) Let A_i, $i \in I$, be nonempty sets and $\succsim$ a binary relation on $A_1 \times A_2 \times \cdots$. The structure $\langle A_1 \times A_2 \times \cdots; \succsim \rangle$ is called an *infinite component structure* iff $\succsim$ satisfies the following five axioms:

1. Weak order.
2. Independence (Definition 1).
3. Restricted solvability (Definition 2).
4. Every strictly bounded standard sequence is finite.
5. At least three components are essential.

(b) If $A_i = A$ for all $i \in I$ then the infinite component structure is called an *I-component structure* and is denoted by $\langle A^I, \succsim \rangle$.

Unfortunately, no general representation can be obtained for infinite component structures. However, it can be shown that representations exist for some substructures defined by additional constraints.

In the following, we consider I-component structures and their representations provided they satisfy the additional properties of *stationarity* and *monotonicity* introduced by Koopmans (1960, 1972).

DEFINITION 6. A binary relation $\succsim$ on A^I is *stationary* iff there exists an $x \in A$, such that

$$(a_1, a_2, ...) \succsim (b_1, b_2, ...) \quad \text{iff} \quad (x, a_1, a_2, ...) \succsim (x, b_1, b_2, ...)$$

for all $(a_1, a_2, ...), (b_1, b_2, ...) \in A^I$.

DEFINITION 7. Let $\succsim$ be an independent weak order on A^I. The relation $\succsim$ is said to satisfy the *monotonicity* condition iff $(a_i \succsim_i b_i)$ for all $i \in I$ implies

$$(a_1, a_2, ...) \succsim (b_1, b_2, ...).$$

Clearly, by independence the element $x \in A$ in the stationarity condition can be replaced by each $y \in A$. Moreover, it should be noted that in the case of finite sequences monotonicity is implied by independence and transitivity.

The first result is a representation of the *ultimately constant* sequences, that is, sequences of the form $(a_1, a_2, ..., a_n, a, a, ...)$ with $a_i, a \in A$.

THEOREM 1 (cf. Koopmans, 1972). *Let $\langle A^I, \succsim \rangle$ be an I-component structure satisfying stationarity and monotonicity. For the set of ultimately sequences with $n \geq 3$ an order-preserving representation is given by*

$$\sum_{i=1}^{n} \lambda^{i-1} u(a_i) + \frac{\lambda^n}{1-\lambda} u(a), \tag{1}$$

with $0 < \lambda < 1$, and u an interval scale on A.

The proof of this theorem follows essentially the lines of Koopmans (1972) and is omitted. Some minor changes are necessary because of the algebraic rather than topologic axioms. Note that the independence condition formulated in Definition 1 is stronger than the one employed by Koopmans. This drawback is, however, compensated for by a much weaker solvability assumption.

In the next theorem an additive representation of all bounded sequences of an I-component structure is developed. As in the case of Theorem 1, a similar result was also proved by Koopmans (1972). This time, however, a major part of the proof, namely case (ii), must be completely rewritten, because Koopmans exploited the topological properties of the reals, namely, their continuity with respect to topological connectedness and separability, in an essential manner. In the following proof this property is completely replaced by the much weaker condition of restricted solvability.

DEFINITION 8. A sequence (a_i) is *bounded*, iff there are $\underline{a}, \bar{a} \in A$, where $\bar{a} \succsim_1 \underline{a}$, such that

$$\bar{a} \succsim_i a_i \succsim_i \underline{a} \qquad \text{for all} \quad i = 1, 2, \ldots.$$

For the set of bounded sequences the following representation theorem holds:

THEOREM 2. *Let $\langle A^I, \succsim \rangle$ be an I-component structure satisfying stationarity and monotonicity. Then, for the set of bounded sequences there exist an interval scale u on A and a unique number $0 < \lambda < 1$, such that*

$$(a_i) \succsim (b_i) \qquad \textit{iff} \quad \sum_{i=1}^{\infty} \lambda^{i-1} u(a_i) \geq \sum_{i=1}^{\infty} \lambda^{i-1} u(b_i). \tag{2}$$

Proof. The interval scale u and the number λ can be obtained from any finite substructure of the I-component structure. Also, $\lambda < 1$ is proved in the same way as in the proof of Theorem 1 (cf. Koopmans, 1972, pp. 87–88). The boundedness of the sequences implies that for all $i = 1, 2, \ldots$,

$$u(\bar{a}) \geq u(a_i) \geq u(\underline{a}) \qquad \text{and} \qquad u(\bar{b}) \geq u(b_i) \geq u(\underline{b}).$$

From $0 < \lambda < 1$ it follows that the two series in (2) are absolutely convergent and, therefore, the sums in (2) are well defined.

We define bounds for any two bounded sequences (a_i) and (b_i) by

$$\underline{g} = \begin{cases} \underline{a}, & \text{if } \underline{b} \succsim \underline{a} \\ \underline{b}, & \text{if } \underline{a} \succ \underline{b}; \end{cases} \qquad \bar{g} = \begin{cases} \bar{a}, & \text{if } \bar{a} \succsim \bar{b} \\ \bar{b}, & \text{if } \bar{b} \succ \bar{a}. \end{cases}$$

It follows that $u(\bar{g}) \geqslant u(\underline{g})$.

Let

$$\Phi[(a_i)] = \sum_{i=1}^{\infty} \lambda^{i-1} u(a_i). \tag{3}$$

(i) Assume first that $\Phi[(a_i)] > \Phi[(b_i)]$ and

$$\Phi[(a_i)] - \Phi[(b_i)] = 3\delta > 0.$$

Consider the two sequences

$$a^n = (a_1, ..., a_n, \underline{g}, \underline{g}, ...) \qquad \text{and} \qquad b^n = (b_1, ..., b_n, \bar{g}, \bar{g}, ...),$$

where n is chosen such that

$$\left(\sum_{i=n+1}^{\infty} \lambda^{i-1} \right) (u(\bar{g}) - u(\underline{g})) = \frac{\lambda^n}{1-\lambda} (u(\bar{g}) - u(\underline{g})) \leqslant \delta.$$

It follows that

$$\Phi[(a_i)] - \Phi(a^n) = \sum_{i=n+1}^{\infty} \lambda^{i-1} [u(a_i) - u(\underline{g})] \leqslant \delta,$$

and

$$\Phi(b^n) - \Phi[(b_i)] \leqslant \delta.$$

This gives

$$\Phi(a^n) - \Phi(b^n) \geqslant \delta > 0,$$

which implies $a^n \succ b^n$ by Theorem 1. Using the monotonicity condition we obtain

$$(a_i) \succsim a^n \succ b^n \succsim (b_i).$$

(ii) Assume next that for two bounded sequences (a_i), (b_i),

$$\Phi[(a_i)] = \Phi[(b_i)] \qquad \text{but} \qquad (b_i) \succ (a_i).$$

holds. There exists an index j such that $b_j \succ_j a_j$, because otherwise monotonicity would imply $(b_i) \precsim (a_i)$.

$$b_j \succ_j a_j \text{ implies } u(b_j) > u(a_j). \tag{4}$$

Let a' denote the sequence $(a_1, ..., a_{j-1}, b_j, a_{j+1}, ...)$; then

$$a' \succ (a_i).$$

The definition of Φ implies

$$\Phi(a') - \Phi[(a_i)] = u(b_j) - u(a_j) > 0$$

by (4). Thus

$$\Phi[(b_i)] = \Phi[(a_i)] < \Phi(a').$$

Case (i) and the assumptions yield $(a_i) \prec (b_i) \prec a'$. By restricted solvability there now exists $c \in A$, such that

$$(a_1, ..., a_{j-1}, c, a_{j+1}, ...) =: a'' \sim (b_i) \succ (a_i). \tag{5}$$

Again by case (i), $\Phi(a'') = \Phi[(b_i)]$ (because $\Phi(a'') < \Phi[(b_i)]$ as well as $\Phi(a'') > \Phi[(b_i)]$ would by case (*i*) contradict (5)).

This implies $\Phi(a'') = \Phi[(a_i)]$. However, direct calculation yields

$$\Phi(a'') - \Phi[(a_i)] = u(c) - u(a_j),$$

hence $u(c) = u(a_j)$ and $c \sim_j a_j$ and $a'' \sim (a_i)$. Therefore $(b_i) \succ a''$, contradicting (5).

Note for later use (proof of Theorem 3) that in this part of the proof the boundedness of the sequences was used only via case (i).

(iii) The last case, $\Phi[(b_i)] > \phi[(a_i)]$, is symmetric to the first. ∎

Finally, we prove a theorem to represent possibly unbounded sequences.

We consider only structures $(A^I, \succsim)$ in which unbounded standard sequences in both directions exist, i.e., there exist $\alpha, \beta \in A$, not $(\alpha \sim_1 \beta)$, and for all $i \in \{0, \pm 1, \pm 2, ...\}$ elements $s_i \in A$ such that $(s_i, \alpha) \sim_{12} (s_{i-1}, \beta)$.

Without losing generality we assume $\alpha \prec_1 \beta$, whence $s_i \prec_1 s_{i+1}$ for all i.

For $a \in A^I$ the function $\Phi(a)$ defined in Eq. (3) can be considered as a particular value of the power series

$$\varphi_a(t) := \sum_{i=1}^{\infty} u(a_i) \cdot t^{i-1}.$$

Clearly, $\Phi(a) = \varphi_a(\lambda)$. The radius of convergence of this power series is given by $r_a := (\limsup_{i \to \infty} |u(a_i)|^{1/i})^{-1}$.

If we consider only $|t| \leqslant 1$ then the *modified radius of convergence* defined by

$$\rho_a := \min(1, r_a)$$

is more suitable for what follows.

If the constant λ of Theorem 2, which is characteristic of the structure $(A^I, \succsim)$, satisfies $\lambda < \rho_a$, then $\Phi(a)$ exists; if $\lambda > \rho_a$, then $\Phi(a)$ does not converge and in the case $\lambda = \rho_a$ the situation may be either way. The function u on A is an interval scale. It is readily verified that ρ_a is unaffected by linear transformations (in fact that is the reason for using ρ_a instead of r_a, because in cases where $r_a > 1$, a transformation $u \mapsto \gamma u + \delta$ with $\delta \neq 0$ reduces the radius of convergence of $\varphi_a(t)$ to 1).

The following lemma gives a formula for the calculation of ρ_a. It is worth mentioning for two reasons: it does not use u and although the sequence $(m_N(a))$ (cf. Definition 9) depends on the particular standard sequence chosen to define it, the left-hand side in Lemma 1 does not.

DEFINITION 9. For $a = (a_i) \in A^I$ and an infinite standard sequence (s_i) in both directions let $m_N(a)$ be the index such that

$$s_{m_N}(a) \precsim_1 a_N \prec_1 s_{m_N(a)+1}.$$

Note that $m_N(a)$ always exists because otherwise the standard sequence would be bounded above or below.

LEMMA 1. *For all unbounded* $a \in A^I$

$$\rho_a = (\limsup_{N \to \infty} |m_N(a)|^{1/N})^{-1}.$$

Proof. By the definition of $m_N(a)$ and the properties of u we have

$$u(s_{m_N(a)}) \leqslant u(a_N) < u(s_{m_N(a)+1}). \tag{6}$$

However, because (s_i) is a standard sequence, $u(s_i)$ is easily evaluated; one proves by induction drawing on Theorem 1:

$$u(s_i) = i\lambda(u(\beta) - u(\alpha)) + u(s_0)) \qquad \text{for} \quad i \in \{0, \pm 1, \pm 2, \ldots\}. \tag{7}$$

With this result (6) yields, with $c = \lambda(u(\beta) - u(\alpha)) > 0$,

$$c \cdot m_N(a) + u(s_0) \leqslant u(a_N) < c \cdot (m_N(a) + 1) + u(s_0).$$

Thus,

$$|u(a_N) - u(s_0)| \begin{cases} \geqslant & c \cdot \min(|m_N(a)|, |m_N(a) + 1|) \\ \leqslant & c \cdot \max(|m_N(a)|, |m_N(a) + 1|). \end{cases} \tag{8}$$

Because of the unboundedness of a the min in Eq. (8) is greater than 0. Therefore $\limsup_{N \to \infty} |m_N(a)|^{1/N} = \limsup_{N \to \infty} |m_N(a) + 1|^{1/N}$; likewise $\limsup_{N \to \infty} |u(a_N) - u(s_0)|^{1/N} = \limsup_{N \to \infty} |u(a_N)|^{1/N}$, yielding the lemma. ∎

To state our main result we need one more concept.

DEFINITION 10. Denote by $n_N(a)$ the index for which it holds that

$$(s_{n_N(a)}, \alpha, ...) \precsim (a_{N+1}, a_{N+2}, ...) \prec (s_{n_N(a)+1}, \alpha, ...).$$

a is of *polynomial growth* if

$$(\limsup_{N\to\infty} |n_N(a)|^{1/N})^{-1} = \rho_a.$$

For a sequence of polynomial growth the behavior of the remainder (i.e., the element $(a_{N+1}, a_{N+2}, ...) \in A^I$) is not extremely different from that of a_N. For example, if $n_N(a) \approx N^k \cdot m_N(a)$ for an arbitrary $k > 0$ then a is of polynomial growth whereas an $a \in A^I$ with $n_N(a) \approx 2^N \cdot m_N(a)$ is not (cf. the relation between ρ_a and $m_N(a)$ derived in Lemma 1).

THEOREM 3. *Let $\langle A^I, \geqslant\rangle$ be an above and below unbounded I-component structure (i.e., infinite standard sequences in both directions exist in $\langle A^I, \geqslant\rangle$) satisfying stationarity and monotonicity. Furthermore, let $(a_i), (b_i) \in A^I$ be of polynomial growth and let λ be in the interior of the circle of convergence of both power series $\Sigma_1^\infty u(a_i)\cdot t^{i-1}$ and $\Sigma_1^\infty u(b_i)\cdot t^{i-1}$, i.e., $\lambda < \rho_a, \rho_b$. Then*

$$(a_i) \succsim (b_i) \qquad \textit{iff} \quad \sum_1^\infty \lambda^{i-1} u(a_i) \geqslant \sum_1^\infty \lambda^{i-1} u(b_i).$$

Proof. (i) Let $\alpha, \beta \in A$ be fixed elements, $\alpha \prec_1 \beta$. Furthermore, let $..., s_{-1}, s_0, s_1, s_2, ... \in A$ be an infinte standard sequence with respect to α, β. Note that existence of (s_i) is guaranteed by restricted solvability and the unboundedness of $\langle A^I, \succsim\rangle$. Assume $a, b \in A^I$ satisfying the assumptions of the theorem and $\Phi(a) > \Phi(b)$, where Φ is defined in Eq. (3). Since the standard sequence (s_i) is finite, for each N there are $n_N(a), n_N(b) \in I$ such that

$$(s_{n_N(a)}, \alpha, ...) \precsim (a_{N+1}, a_{N+2}, ...) \prec (s_{n_N(a)+1}, \alpha, ...)$$

and mutatis mutandis for b.

By solvability we find $\xi_N, \eta_N \in A$ such that $(\xi_N, \alpha, ...) \sim (a_{N+1}, a_{N+2}, ...)$ and $(\eta_N, \alpha, ...) \sim (b_{N+1}, b_{N+2}, ...)$. From stationarity we infer

$$\xi^N := (a_1, ..., a_N, \xi_N, \alpha, \alpha, ...) \sim a, \qquad \eta^N := (b_1, ..., b_N, \eta_N, \alpha, \alpha, ...) \sim b. \tag{9}$$

However, ξ^N and η^N are ultimately constant sequences. Thus, if we can show

$$\Phi(\xi^N) > \Phi(\eta^N), \tag{10}$$

then by Theorem 1 we can conclude $\xi^N \succ \eta^N$, which by (9) yields $a \succ b$. To prove (10), we observe

$$\Phi(\xi^N) = \sum_{i=1}^N u(a_i)\cdot\lambda^{i-1} + \lambda^N \cdot u(\xi_N). \tag{11}$$

From the definition of ξ_N it follows that $u(s_{n_N(a)}) \leqslant u(\xi_N) < u(s_{n_N(a)+1})$. Inserting Eq. (7) in this estimation yields

$$u(\xi_N) = c \cdot n_N(a) + O(1), \tag{12}$$

where $O(1)$ is some bounded function and c a constant. Since λ is in the interior of the circle of convergence, we have

$$\lambda < (\limsup_{n \to \infty} |u(a_n)|^{1/n})^{-1}. \tag{13}$$

Moreover by Lemma 1, $\limsup_{n \to \infty} |u(a_n)|^{1/n} = \limsup_{N \to \infty} (m_N(a))^{1/N}$. By polynomial growth of a we derive, from Eq. (13), $\lambda < (\limsup_{N \to \infty} (n_N(a))^{1/N})^{-1}$ or $\lambda (n_N(a))^{1/N} \leqslant \gamma < 1$ for all but finitely many N, whence $\lambda^N n_N(a) \to 0$. Inserting Eq. (12) into Eq. (11) and using the result just proved yield $\Phi(\xi^N) \to \Phi(a)$ for $N \to \infty$. Equivalently we find $\Phi(\eta^N) \to \Phi(b)$ for $N \to \infty$.

Now, let N be large enough. Then the assumption $\Phi(a) > \Phi(b)$ yields Eq. (10) and by Theorem 1, $a \succ b$.

(ii) The case $\Phi(a) = \Phi(b)$ but $a \succ b$ is analogous to the corresponding proof in Theorem 2. ∎

It has already been mentioned that in our theorems independence conditions stronger than those used by Koopmans (1972) are assumed. The reason lies in the strong topological requirements entailed in Koopmans' axioms. He formulates and proves his theorems only for sets A which are connected subsets of the n-dimensional Euclidean space (cf. Koopmans, 1972, Postulate 1, p. 81). It is easy to reformulate our theorems with independence conditions using only sets M such that $|M| \leqslant 2$ or $|I - M| \leqslant 2$ (cf. Definition 1), which is almost as weak as in Koopmans (1972), if one is willing to stipulate a stronger solvability condition.

Another point of interest is the gap between Theorems 2 and 3. If A is bounded in the sense that all standard sequences are finite but A does not possess a maximal or minimal element, then $\Phi(x)$ is convergent for all $x \in A^I$, but we cannot draw on Theorem 2 for a representation of $\succsim$ because some sequences will have no bounds $\bar{a}$, $\underline{a}$. We conjecture that an artificially defined maximal (minimal) element adjoined to A and a suitable extension of $\succsim$ can reduce this case to Theorem 2, but at present an extension theorem applicable in this case seems not to be available. Koopmans (1972) suggests a condition which is strong enough to ensure the existence of a constant sequence equivalent to $a \in A^I$ with convergent $\Phi(a)$. Obviously, in this case the proof technique of Theorem 2 yields the desired representation theorem.

Finally, the case with convergent $\Phi(a)$ but λ on the boundary of the circle of convergence of the power series $\sum_{i=1}^{\infty} u(a_i) \cdot t^{i-1}$ is not covered by Theorem 3 and remains as un unsolved problem.

CONCLUDING REMARKS

With the proofs provided in this paper the results of Koopmans (1972), concerning the representation of infinite component sequences, have been adapted to the common and quite general framework of algebraic measurement theory and have also been extended.

Unfortunately, infinite component sequences have been fairly neglected in measurement theory, although they are interesting not only for modelling dynamic decision making as intended by Koopmans. Interpreting the index of the components as a time index allows one to go beyond this example and consider other dynamic relations as well. Moreover, such structures can be related to standard linear system theory (see Hübner, 1989a). For certain types of linear systems it is possible to measure their input while simultaneously identifying their input–output relation (cf. Hübner, 1989a, 1989b).

Testing whether or not such infinte structures are empirically valid faces the same problems as other theorems formulated for infinite sets. Such theories are idealizations and in this form not empirically testable. Empirical applications would have to rely on finite-components structures like the n-component structures considered by Krantz *et al.* (1971) and on standard methods for testing axioms of conjoint measurement theory. Krantz *et al.* (1971) formulated a stationarity condition for n-component structures which is extended here for infinite component structures. They also provide another interesting and less strong condition, a *standard sequence condition*. One way to relate n-component and infinite component structures is by assuming that for the latter only n components are essential.

Testing the additional conditions of Theorem 3 is also not possible in the strict sense. Both polynomial growth and $\lambda < \rho_a$ are asymptotic concepts that cannot be verified of falsified by data. One can, however, determine $m_N(a)$ and $n_N(a)$ for some indices N and try to "guess" the behavior of $|m_N(a)|^{1/N}$ and $|n_N(a)|^{1/N}$. In this respect Lemma 1 is useful.

However, it should be pointed out that for obtaining general theoretical derivations concerning the representation of dynamic behavior, n-component structures are too restrictive because they require fixing the number of components. This is, for instance, insufficient for connecting conjoint structures with linear system theory.

REFERENCES

DEBREU, G. (1960). Topological methods in cardinal utility theory. In K. J. Arrow, S. Karlin, & P. Suppes (Eds.), *Mathematical methods in the social sciences*. Stanford, CA: Stanford Univ. Press.

DIAMOND, P. A. (1965). The evaluation of infinite utility streams. *Econometrica*, **33**, 170–177.

HÜBNER, R. (1989a). The representation of dynamic structures. In E. E. Roskam (Ed.), *Mathematical psychology in progress*. New York: Springer-Verlag.

HÜBNER, R. (1989b). Ein dynamisches Modell zur Lautheitsadaption. *Zeitschrift für Psychologie*, **197**, 97–107.

KOOPMANS, T. C. (1960). Stationary ordinal utility and impatience. *Econometrica*, **28**, 287–309.

KOOPMANS, T. C. (1972). Representation of preference orderings over time. In C. B. McGuire & R. Radner (Eds.), *Decision and organization*. Amsterdam/London: North-Holland.

KOOPMANS, T. C., DIAMOND, P. A., & WILLIAMSON, R. E. (1964). Stationary utility and time perspective. *Econometrica*, **32**, 82–100.

KRANTZ, D. H., LUCE, R. D., SUPPES, P., & TVERSKY, A. (1971). *Foundations of measurement*, Vol I. New York: Academic Press.

WAKKER, P. (1988). The algebraic versus the topological approach to additive representations. *Journal of Mathematical Psychology*, **32**, 421–435.

RECEIVED: May 20, 1992